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THE INERTIAL WAVE FREQUENCY SPECTRUM IN A CYLINDRICALLY CONFINED, INVISCID, INCOMPRESSIBLE TWO COMPONENT LIQUID

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A theoretical and experimental study is made of the phenomenon indicated in the title. It is shown that for inertial waves, just as for gravity waves, there are discontinuities in the tangential particle velocities at the interface in a real, two-component liquid, a fact implying the existence of a vortex sheet. For the case where the two liquids completely fill the cylinder, other results are obtained that are analogous to those for gravity waves. In particular, if the liquids are nearly of the same density, there are two sets of frequencies, one set characterizing oscillations of the liquid mass as a whole, the other set characterizing very low frequency oscillations at the interface. For the case in which the two liquids have markedly different densities, there are again two sets of frequencies, one set characterizing oscillations of the inner liquid as though the outer liquid were a solid mass, the other set characterizing oscillations of the outer liquid as though the inner liquid were absent. For the general case, physical interpretations are difficult; hence, a table of frequencies versus composition is given.

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TABLE OF CONTENTS

LIST OF SYMBOLS

a	inner radius of free surface
b	radius of surface separating the two liquids
c	outer radius of cavity
d	one-half cylinder height
i	$\sqrt{-1}$
J	Bessel function of first kind
K_1	$\Omega^2 a/S^2\ [1 + (2\Omega/S)^2]$
K_2	$2\Omega i/S$
K_3	$((2j + 1)\ \pi i/2d)/\sqrt{1 + (2\Omega/S)^2}$
p	pressure
r,θ,z	cylindrical coordinates
S	frequency
t	time
$\underline{u}$	perturbation velocity
Y	Bessel function of second kind
ρ	density
Ω	angular speed of liquid

I. INTRODUCTION

Lamb[1*] notes some interesting facts about gravity wave motion in inviscid, incompressible, superposed liquids of different densities, e.g., the diminution of the speed of a wave from the value it would have in a one-component liquid, the long natural period of oscillation of the interface of two liquids of nearly equal density, the mathematically possible but experimentally unobserved "breaking" of the waves at the interface, the independent oscillations of the two liquids if the densities are markedly different, and the presence of a vortex sheet at the interface. This note investigates mathematically and experimentally whether that class of internal waves now commonly termed inertial waves (Bjerkness[2], and Fultz[3]) exhibit such phenomena when confined in a right circular cylinder.

II. ANALYSIS

A. Mathematical

1. The Governing Equations. Consider Figure 1. Following Stewartson[3], we initially assume that the inviscid, incompressible liquids are spinning uniformly as a rigid body, that $\Omega^2 b^2$ and $\Omega^2 c^2$ are much greater than dg, and that $\rho_2 > \rho_1$. Then in this steady state, we have:

$$0 = - \nabla \left[\frac{P_{10}}{\rho_1} - \frac{\Omega^2}{2} (r^2 - a^2) \right], \quad a < r < b \tag{1}$$

and

$$0 = - \nabla \left[\frac{P_{20}}{\rho_2} - \frac{\rho_1}{\rho_2} \frac{\Omega^2}{2} (b^2 - a^2) - \frac{\Omega^2}{2} (r^2 - b^2) \right], \quad b < r < c \tag{2}$$

where P_{10} and P_{20} are the "static" pressures in the uniformly spinning liquids, and where the coordinate system has the angular velocity $\underline{\Omega} = (0,0,\Omega)$.

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*References are listed on page 21.

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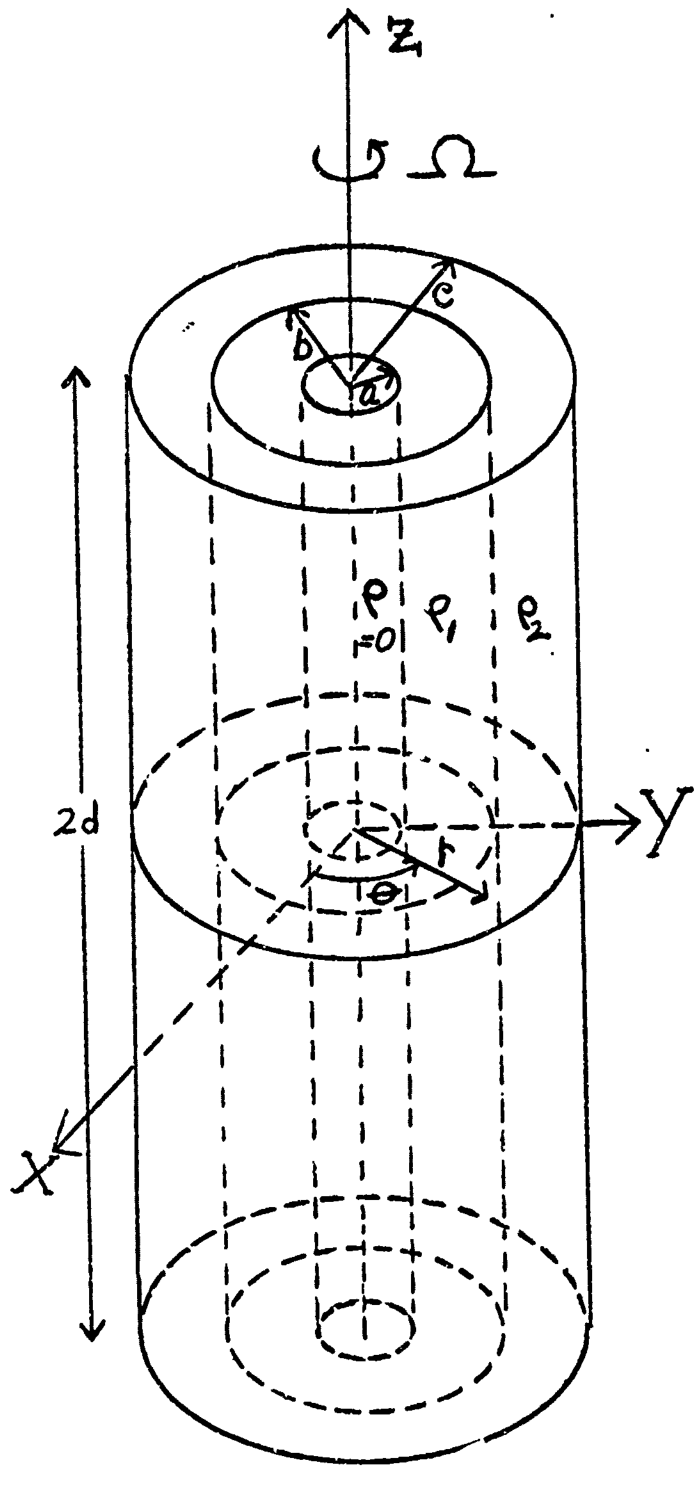

Figure 1. Definition of Coordinate System

After a momentary disturbance of the container that generates a "perturbation" velocity $\underline{u} = (u,v,w)$, the linearized Euler equations in the two regions are:

$$\frac{\partial}{\partial t}\underline{u}_1 + 2\Omega \times \underline{u}_1 = -\nabla\left[\frac{P_1}{\rho_1} - \frac{\Omega^2}{2}(r^2 - a^2)\right] \tag{3}$$

and

$$\frac{\partial}{\partial t}\underline{u}_2 + 2\underline{\Omega} \times \underline{u}_2 = -\nabla\left[\frac{P_2}{\rho_2} - \frac{\rho_1}{\rho_2}\frac{\Omega^2}{2}(b^2 - a^2) - \frac{\Omega^2}{2}(r^2 - b^2)\right] \tag{4}$$

where P_1, $\underline{u}_1$ and P_2, $\underline{u}_2$ are the two pressures and perturbation velocities in the two regions after the disturbance. Subtracting (1) from (3) and (2) from (4), we have

$$\frac{\partial}{\partial t}\underline{u}_k + 2\underline{\Omega} \times \underline{u}_k = -\nabla\left(\frac{P_k - P_{ko}}{\rho_k}\right) \equiv -\nabla\frac{P_k'}{\rho_k}, \quad k = 1,2 \tag{5}$$

where P_1' and P_2' may properly be termed perturbation pressures.

Again following Stewartson[4], we seek normal mode solutions of equation (5). Hence, letting all time-dependent terms have the same functional e^{st} dependence, one gets (Miles[5], 1959):

$$\underline{u}_k = \left[\frac{2\underline{\Omega}}{S} \times - 1 - \frac{2\underline{\Omega}}{S}\frac{2\underline{\Omega}}{S}\cdot\right]\frac{\nabla P_k'}{S\rho_k\ [1 + (2\Omega/S)^2]} \tag{6}$$

From the continuity equation $\nabla \cdot \underline{u}_k = 0$, one gets:

$$\nabla^2 P_k' + \left(\frac{2\underline{\Omega}}{S}\cdot\nabla\right)^2 P_k' = 0 \tag{7}$$

We solve equation (7) by the usual separation of variables process, getting:

$$P_k'(r,\theta,z) = \sum_{m_k=1} e^{im_k\theta} \left[A_k J_{m_k}\left(\frac{\lambda_k}{d} r\sqrt{1+\left(\frac{2\Omega}{S}\right)^2}\right) + B_k Y_{m_k}\left(\frac{\lambda_k}{d} r\sqrt{1+\left(\frac{2\Omega}{S}\right)^2}\right)\right] \cdot$$

$$\cdot \left[C_k \cos h \frac{\lambda_k}{d} z + D_k \sin h \frac{\lambda_k}{d} z\right] \tag{8}$$

where the J's and Y's are Bessel functions of the first and second kinds, respectively, where the λ's and m's are separation constants, and where A_k, B_k, C_k, D_k are integration constants.

2. <u>The Frequency Equation</u>. After the disturbance, the container has only the angular velocity (o,o,Ω); hence, the boundary condition on all solid surfaces is $\underline{u} \cdot \underline{n} = 0$, where $\underline{n}$ is an outward directed unit normal. On the end face, then, we have:

$$\left.\frac{\partial P_k'}{\partial z}\right]_{z = \pm d} = 0 \tag{9}$$

from which there follows $C_k = 0$, and

$$\lambda_1 = \lambda_2 = (2j+1)\frac{\pi i}{2}, \; i = \sqrt{-1}, \; j = 0,1,2,\ldots \tag{10}$$

Hence, redefining our constants, we rewrite (8) as:

$$P_k' = \sum_{j=0} \sum_{m_k=1} e^{im_k\theta} \left[A_k' J_{m_k}(K_3 r) + B_k' Y_{m_k}(K_3 r)\right] \sin (2j+1)\frac{\pi z}{2d} \tag{11}$$

where

$$K_3 \equiv (2j+1)\frac{\pi i}{2d}\sqrt{1+\left(\frac{2\Omega}{S}\right)^2} \tag{12}$$

On the lateral surface $r = c$, the condition $\underline{u} \cdot \underline{n} = 0$ gives:

$$\left[\frac{\partial P'_2}{\partial r} + \frac{2\Omega}{Sr}\frac{\partial P'_2}{\partial \theta}\right]_{r=c} = 0$$

$$= A'_2\left[K_3 J'_{m_2}(K_3 c) + K_2 \frac{m_2}{c} J_{m_2}(K_3 c)\right]$$

$$+ B'_2\left[K_3 Y'_{m_2}(K_3 c) + K_2 \frac{m_2}{c} Y_{m_2}(K_3 c)\right] \quad (13)$$

where

$$K_2 \equiv \frac{2\Omega i}{S} \quad (14)$$

In section B we outline how we search for the liquid eigenfrequencies experimentally by adjusting certain physical parameters until the gyroscope undamps. Inferring that this undamping is due to resonance between one of the liquid eigenfrequencies and the nutational frequency of the gyroscope (see Stewartson[4], 1957), one can determine that particular eigenfrequency. Though the validity of this procedure is well documented, it nevertheless affords a determination of only those modes for which $m_2 = 1$, for one can show that coupling between the liqud and shell motions occurs only for that value of m_2. Hence, the form of equation (13) pertinent for this study is:

$$A'_2\left[K_3 J'_1(K_3 c) + \frac{K_2}{c} J_1(K_3 c)\right]$$

$$+ B'_2\left[K_3 Y'_1(K_3 c) + \frac{K_2}{c} Y_1(k_3 c)\right] = 0 \quad (15)$$

On the inner free surface (at r = a), the kinematical boundary condition is $\frac{DF}{Dt} = 0$, where $F \equiv r - a - \eta(\theta,z,t) = 0$, and $\eta(\theta,z,t)$ is the free surface elevation. Hence, to first order, the kinematical boundary condition is

$$u_1 - \frac{\partial\eta}{\partial t}\Big]_{r=a+\eta} = 0 = u_1 - S\eta\Big]_{r=a+\eta} \tag{16}$$

The dynamic boundary condition on the inner free surface is $P_1 = 0$. In terms of P'_1, this is:

$$P'_1\Big]_{r=a+\eta} = -P_{10}\Big]_{r=a+\eta} \doteq -\rho_1 \Omega^2 a\eta. \quad \text{Hence:}$$

$$SP'_1 + \rho_1 \Omega^2 a u_1\Big]_{r=a+\eta} = 0 \tag{17}$$

which, in terms of (11), is:

$$A'_1\left[J_{m_1}(K_3 a) - K_1\left\{K_3 J'_{m_1}(K_3 a) + K_2 \frac{m_1}{a} J_{m_1}(K_3 a)\right\}\right]$$

$$+ B'_1\left[Y_{m_1}(K_3 a) - K_1\left\{K_3 Y'_{m_1}(K_3 a) + K_2 \frac{m_1}{a} Y_{m_1}(K_3 a)\right\}\right] = 0 \tag{18}$$

where

$$K_1 \equiv \frac{\Omega^2 a}{S^2\left[1 + \left(\frac{2\Omega}{S}\right)^2\right]} \tag{19}$$

At the interface $r = b$, continuity of the pressure and normal velocity yield two dynamic boundary conditions[6]. From the continuity of the normal velocity we have:

$$u_1\Big]_{r=b+\eta} = u_2\Big]_{r=b+\eta}, \text{ or:}$$

$$\sum_{m_1=1} \frac{e^{im_1\theta}}{\rho_1} \Big[K_3 \left\{A_1' J_{m_1}' (K_3 b) + B_1' Y_{m_1}' (K_3 b)\right\}$$

$$+ K_2 \frac{m_1}{b} \left\{A_1' J_{m_1} (K_3 b) + B_1' Y_{m_1} (K b)\right\} \Big]$$

$$= \frac{e^{i\theta}}{\rho_2} \Big[K_3 \left\{A_2' J_1' (K_3 b) + B_2' Y_1' (K_3 b)\right\}$$

$$+ K_2 \frac{1}{b} \left\{A_2' J_1 (K_3 b) + B_2' Y_1 (K_3 b)\right\} \Big] \tag{20}$$

Hence, $m_1 = 1$.

From the equality of pressures at the interface, we have $\left(P_1\right)_{r=b+\eta} = \left(P_2\right)_{r=b+\eta}$, or $\left(P_1' + P_{10}\right)_{r=b+\eta} = \left(P_2' + P_{20}\right)_{r=b+\eta}$. Hence, to first order in η, we have: $(P_1'-P_2')_{r=b+\eta} = b\eta\Omega^2(\rho_2 - \rho_1)$, from which there follows: $\left[SP_1' + \rho_1\Omega^2 bu_1 = SP_2' + \rho_2\Omega^2 bu_2\right]_{r=b+\eta}$. Substituting for P_1', P_2', u_1, and u_2, we have:

$$A_1' \Big[J_1 (K_3 b) - K_1 \frac{b}{a} \left\{K_3 J_1' (K_3 b) + \frac{K_2}{b} J_1 (K_3 b)\right\}\Big]$$

$$+ B_1' \Big[Y_1 (K_3 b) - K_1 \frac{b}{a} \left\{K_3 Y_1' (K_3 b) + \frac{K_2}{b} Y_1 (K_3 b)\right\}\Big]$$

$$= A_2' \Big[J_1 (K_3 b) - K_1 \frac{b}{a} \left\{K_3 J_1' (K_3 b) + \frac{K_2}{b} J_1 (K_3 b)\right\}\Big]$$

$$+ B_2' \Big[Y_1 (K_3 b) - K_1 \frac{b}{a} \left\{K_3 Y_1' (K_3 b) + \frac{K_2}{b} Y_1 (K_3 b)\right\}\Big] \tag{21}$$

Equations (15), (18), (20), and (21) are a set of four homogeneous equations for the determination of A_1', B_1', A_2', and B_2'. For convenience, we re-write these four equations in the form

$$A_1' \cdot 0 + B_1' \cdot 0 + A_2'\ell_3(c) + B_2'\ell_4(c) = 0 \qquad (22)$$

$$A_1'\ell_1(a) + B_1'\ell_2(a) + A_2' \cdot 0 + B_2' \cdot 0 = 0 \qquad (23)$$

$$A_1'\rho_2\ell_3(b) + B_1'\rho_2\ell_4(b) - A_2'\rho_1\ell_3(b) - B_2'\rho_1\ell_4(b) = 0 \qquad (24)$$

$$A_1'\ell_1(b) + B_1'\ell_2(b) - A_2'\ell_1(b) - B_2'\ell_2(b) = 0 \qquad (25)$$

where the definitions of the several ℓ's are obvious.

From the above set of equations one can express P_1' and P_2' in terms of A_1'. Then, from $v_1 = \underline{u}_1 \cdot \underline{i}_\theta = \dfrac{1}{\rho_1 S\left[1 + \left(\frac{2\Omega}{S}\right)^2\right]}\left[\dfrac{2\Omega}{S}\dfrac{\partial P_1'}{\partial r} - \dfrac{1}{r}\dfrac{\partial P_1'}{\partial \theta}\right]$ and

$v_2 = \underline{u}_2 \cdot \underline{i}_\theta = \dfrac{1}{\rho_2 S\left[1 + \left(\frac{2\Omega}{S}\right)^2\right]}\left[\dfrac{2\Omega}{S}\dfrac{\partial P_2'}{\partial r} - \dfrac{1}{r}\dfrac{\partial P_2'}{\partial \theta}\right]$, one can show that

$v_1\big]_{r=b} \neq v_2\big]_{r=b}$. Hence, as is the case for gravity waves in superposed liquids[1], one infers that for inertial waves in a real liquid there would be a vortex sheet at the interface.

Equations (22) - (25) are a homogeneous set, and a necessary condition that A_1', B_1', A_2', and B_2' be non-zero is that the determinant of the coefficients be zero. Hence:

$$\begin{vmatrix} 0 & 0 & \ell_3(c) & \ell_4(c) \\ \ell_1(a) & \ell_2(a) & 0 & 0 \\ \rho_2\ell_3(b) & \rho_2\ell_4(b) & -\rho_1\ell_3(b) & -\rho_1\ell_4(b) \\ \ell_1(b) & \ell_2(b) & -\ell_1(b) & -\ell_2(b) \end{vmatrix} = 0 \tag{26}$$

which is the frequency equation.

We first consider a special case of (26). Let the two liquids completely fill the cylinder. Then a = 0 and equation (26) reduces to

$$\begin{vmatrix} 0 & \ell_3(c) & \ell_4(c) \\ \rho_2\ell_3(b) & -\rho_1\ell_3(b) & -\rho_1\ell_4(b) \\ \ell_1(b) & -\ell_1(b) & -\ell_2(b) \end{vmatrix} = 0 = -(\rho_2 - \rho_1)\ \ell_1(b)\ell_3(b)\ell_4(c) - \ell_3(c)\left[\rho_1\ell_1(b)\ell_4(b) - \rho_2\ell_3(b)\ell_2(b)\right] \tag{27}$$

We now consider two special cases of equation (27).

Case I: $\rho_2 \doteqdot \rho_1$

Then, from (27), we have:

$$\ell_3(c) \doteqdot K_3\ J_1'(K_3c) + \frac{K_2}{c} J_1(K_3c) = 0 \tag{28}$$

which is Stewartson's inertial wave frequency equation for a one-component liquid completely filling a cylinder of radius c.

Alternatively, from (27) we have:

$$\ell_4(b)\ell_1(b) - \ell_3(b)\ell_2(b) \equiv J_1(K_3b)\left[K_3Y_1'(K_3b) + \frac{K_2}{b} Y_1(K_3b)\right] - Y_1(K_3b)\left[K_3\ J_1'(K_3b) + \frac{K_2}{b} J_1(K_3b)\right] = 0 \tag{29}$$

Using the asymptotic expansions for the Bessel functions, one can show that (29) is satisfied if K_3 is very large. However, a large K_3

implies a small δ, a result establishing the existence of low frequency waves at the interface.

Hence, we've shown that, if $\rho_1 \doteq \rho_2$, there are two sets of inertial wave frequencies, one set for the liquid as a whole, the other for the oscillations at the interface. A similar result holds for gravity waves.

Case II: $\rho_2 >> \rho_1$

Then, from (27), we have:

$$\ell_3(b) \equiv K_3 J_1'(K_3 b) + \frac{K_2}{b} J_1(K_3 b) = 0 \tag{30}$$

This would be the Stewartson frequency equation for the inner liquid if the outer liquid were a solid mass.

Alternatively, from (27), we have:

$$\ell_1(b)\ell_4(c) - \ell_3(c)\ell_2(b) = 0$$

$$= \left[J_1(K_3 b) - K_1(b)\left\{K_3 J_1'(K_3 b) + \frac{K_2}{b} J_1(K_3 b)\right\}\right]\left[K_3 Y_1'(K_3 c) + \frac{K_2}{c} Y_1(K_3 c)\right]$$

$$- \left[K_3 J_1'(K_3 c) + \frac{K_2}{c} J_1(K_3 c)\right]\left[Y_1(K_3 b) - K_1(b)\left\{K_3 Y_1'(K_3 b) + \frac{K_2}{b} Y_1(K_3 b)\right\}\right] \tag{31}$$

This is the Stewartson frequency equation for a one component liquid, with a free surface at $r = b$, in a cylinder of radius c.

Hence, we've established that, if $\rho_2 >> \rho_1$, there are also two sets of inertial wave frequencies, one characterizing the inner liquid as though the outer were solid; the other characterizing the outer liquid as though the inner were absent. A similar result holds for gravity waves.

For the general case, where $a \neq 0$, and ρ_1 and ρ_2 are neither nearly equal nor far removed, equation (26) does not easily lend itself to such

physical interpretations. Hence, we merely give a table of theoretically and experimentally determined frequencies versus composition for a particular density ratio*.

Table I.

$\frac{\rho_1}{\rho_2}$	$\frac{c}{d}$	$\frac{b}{d}$	$\frac{a}{d}$	$\frac{S}{i\Omega}$ (Theory)	$\frac{S}{i\Omega}$ (Experimental)
.8	3.127	2.708	0	- .931	- .936
.8	3.127	2.211	0	- .927	- .934
.8	3.127	1.564	0	- .938	- .942
.8	3.127	2.752	.989	- .926	- .924
.8	3.127	2.319	.989	- .914	- .920
.8	3.127	1.783	.989	- .931	- .926

B. Experimental

Two immiscible liquids are placed in a cylindrical cavity inside a gyroscope. Details of the gyroscope, supporting apparatus, and the experimental procedure for determining the liquid eigenfrequencies by resonating one of them with the nutational frequency of the gyroscope, are profusely documented elsewhere[4]. We merely note here that the inertial waves can create an asymmetrical pressure distribution in the cavity; and the above-mentioned state of resonance merely ensures that the asymmetrical pressure distribution will "keep step" with the nutational motion, thus causing an overturning effect that results in a growth of the nutational component of the motion of the gyroscope. Since the effect involves integrals like

$$\int_0^{2\pi} e^{im\theta} (\cos\theta, \sin\theta)\, d\theta$$

that are zero unless $m = 1$, only those modes of oscillation for which

*See Section B.

$m = 1$ couple with the shell motion. Hence, it is only such modes that we can find with the gyroscope.

Since viscosity undoubtedly has an effect, it was essential that the two liquids of different density have small, equal, kinematic viscosities. Fortunately, the Dow-Corning Company has just developed a light oil that has a kinematic viscosity (one centistoke) equal to that of water, yet with a specific gravity of 0.8. All of the experiments were run using water and this oil, and the results are given in Table I.

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